JOURNAL OF APPROXIMATION THEORY **53**, 365–366 (1988)

# Note

# On the Size of the Coefficients of Rational Functions Approximating Powers

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*Communicated by Oved Shisha*

Received March 31, 1986; revised February 22, 1988

1. The following was proved in [1]:

THEOREM A. *Let* $a_0, a_1, ..., a_n, b_0, b_1, ..., b_n$ $(n \geqslant 1)$ *be integers so that*

$$\max_{-1 \leqslant x \leqslant 1} \left| |x| - \left[ \sum_{k=0}^{n} a_k x^k \Big/ \sum_{k=0}^{n} b_k x^k \right] \right| < \varepsilon, \qquad 0 < \varepsilon < \tfrac{1}{4}.$$

*Then* $\max(|a_0|, ..., |a_n|, |b_0|, ..., |b_n|) > (5\sqrt{\varepsilon})^{-1}$.

We prove here an analogous but stronger result for the function $\sqrt{x}$, namely,

THEOREM B. *Let* $a_0, a_1, ..., a_n, b_0, b_1, ..., b_n$ $(n \geqslant 1)$ *be complex numbers where either* $a_0 = 0$ *or* $|a_0| \geqslant 1$ *and where* $|b_0| \geqslant 1$, *so that*

$$\max_{0 \leqslant x \leqslant 1} \left| \sqrt{x} - \left[ \sum_{k=0}^{n} a_k x^k \Big/ \sum_{k=0}^{n} b_k x^k \right] \right| < \varepsilon, \qquad 0 < \varepsilon \leqslant \tfrac{1}{4}.$$

*Then*

$$\mu = \max(|a_0|, ..., |a_n|, |b_0|, ..., |b_n|) > (7\varepsilon)^{-1}.$$

*Proof.* If $a_0 \neq 0$, then $|a_0/b_0| < \varepsilon$ and hence

$$\mu \geqslant |b_0| > |a_0|/\varepsilon \geqslant 1/\varepsilon > (7\varepsilon)^{-1}.$$

Suppose $a_0 = 0$. Then

$$\left| \sum_{k=0}^{n} a_k (4\varepsilon^2)^k \right| \leqslant \sum_{k=1}^{n} |a_k| (4\varepsilon^2)^k < 4\varepsilon^2 \mu \sum_{k=0}^{\infty} (4\varepsilon^2)^k \leqslant 4\varepsilon^2 \mu \sum_{k=0}^{\infty} 4^{-k} = 16\varepsilon^2 \mu/3,$$



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$$\left|\sum_{k=0}^{n} b_k(4\varepsilon^2)^k\right| \geqslant |b_0| - \sum_{k=1}^{n} |b_k|(4\varepsilon^2)^k$$

$$> 1 - 4\varepsilon^2\mu \sum_{k=0}^{\infty} (4\varepsilon^2)^k > 1 - 4\varepsilon^2\mu \sum_{k=0}^{\infty} 4^{-k} = 1 - (16\varepsilon^2\mu/3).$$

If $1 - (16\varepsilon^2\mu/3) \leqslant 0$, then $\mu \geqslant 3/(16\varepsilon^2) > (7\varepsilon)^{-1}$. Otherwise,

$$\varepsilon > \left|\sqrt{4\varepsilon^2} - \left[\sum_{k=0}^{n} a_k(4\varepsilon^2)^k \Big/ \sum_{k=0}^{n} b_k(4\varepsilon^2)^k\right]\right|$$

$$\geqslant 2\varepsilon - \left[\left|\sum_{k=0}^{n} a_k(4\varepsilon^2)^k\right| \Big/ \left|\sum_{k=0}^{n} b_k(4\varepsilon^2)^k\right|\right]$$

$$> 2\varepsilon - [16\varepsilon^2\mu/(3 - 16\varepsilon^2\mu)]$$

and hence

$$16\varepsilon\mu \cdot \tfrac{5}{4} \geqslant 16\varepsilon\mu(1+\varepsilon) > 3,$$

$$\mu > 3/(20\varepsilon) > (7\varepsilon)^{-1}.$$

2. Similarly, we have

THEOREM C. *Let $\alpha \geqslant 1$ and let $b_0, ..., b_n$ $(n \geqslant 1)$ be complex numbers such that*

$$\max_{0 \leqslant x \leqslant 1} \left|x^\alpha - \left(\sum_{k=0}^{n} b_k x^k\right)^{-1}\right| < \varepsilon, \qquad 0 < \varepsilon \leqslant \tfrac{1}{2}.$$

*Then*

$$\mu = \max(|b_0|, ..., |b_n|) > (4\varepsilon)^{-1}.$$

*Proof.*

$$\varepsilon > \left|\left(\sum_{k=0}^{n} b_k \varepsilon^k\right)^{-1} - \varepsilon^\alpha\right| \geqslant \left|\sum_{k=0}^{n} b_k \varepsilon^k\right|^{-1} - \varepsilon > \left[\mu \sum_{k=0}^{\infty} 2^{-k}\right]^{-1} - \varepsilon = (2\mu)^{-1} - \varepsilon.$$

Hence $\mu > (4\varepsilon)^{-1}$.

## REFERENCE

1. L. B. O. FERGUSON AND J. SZABADOS, The growth of coefficients of rational functions with integral coefficients which approximate $|x|$, *Studia Sci. Math. Hungar.* **18** (1983), 73–77.